Continuous variable remote state preparation

Z. Kurucz,[1] P. Adam,[1, 2] Z. Kis,[1] and J. Janszky[1, 2]

[1]*Department of Nonlinear and Quantum Optics, Research Institute for Solid State Physics and Optics,*
Hungarian Academy of Sciences, PO Box 49 H-1525 Budapest, Hungary
[2]*Research Group for Nonlinear Optics, Hungarian Academy of Sciences,*
and Institute of Physics, University of Pécs, Ifjúság út 6. H-7624 Pécs, Hungary

We extend exact deterministic remote state preparation (RSP) with minimal classical communication to quantum systems of continuous variables. We show that, in principle, it is possible to remotely prepare states of an ensemble that is parameterized by infinitely many real numbers, i.e., by a real function, while the classical communication cost is one real number only. We demonstrate continuous variable RSP in three examples using (i) quadrature measurement and phase space displacement operations, (ii) measurement of the optical phase and unitaries shifting the same, and (iii) photon counting and photon number shift.

I. INTRODUCTION

In remote state preparation (RSP), quantum communication is combined with quantum state engineering. The sender (Alice), having a classical description of the state, prepares a physical instance of its at the receiver's distant location (at Bob's laboratory) using previously shared entanglement and classical communication as resources. A plausible protocol implementing this task is the one in which Alice prepares the state locally in her lab and teleports it to Bob [1]. This protocol consumes 1 ebit of entanglement and communicates 2 bits of classical information per each qubit prepared. However, it may be preferable to avoid teleportation in certain situations, especially, if it is difficult to implement the Bell-type measurements essential for the protocol.

Recently, several non-teleportation-based RSP protocols have been developed [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. They exploit the fact that Alice has complete classical knowledge of the state to be prepared. This gives the possibility to trade off the resources. For example, an alternative for Alice is to tell everything to Bob so that he can prepare the state locally in his lab. This method requires no prior entanglement but the transfer of two real numbers, that is, infinitely many bits of classical information. In the other limit posed by causality, it is possible to communicate an arbitrary qubit using one classical bit only. The asymptotic protocol II of Ref. [2] achieves this rate utilizing 1 ebit per transmitted qubit. Besides these two extreme cases, trade-off among resources has been widely investigated in the literature [3, 4, 5, 6, 7].

A conceptually different kind of RSP was introduced in Ref. [8]. This achieves a trade-off from another point of view: resources can also be cut down if the input is restricted to an ensemble of special pure qubit states. For example, equatorial or polar great circles on the Bloch sphere can be prepared remotely with probability 1 using one ebit of entanglement and 1 bit of classical communication. The protocol does not suffer from the difficulties associated with the Bell measurement of quantum teleportation because Alice performs a simple onepartite measurement on her half of the entangled pair, projecting its spin along some well-defined direction. Furthermore, instead of the four unitaries of the teleportation protocol, Bob applies either the identity or σ_z.

This kind of RSP has recently been generalized to the N dimensional case [9, 10, 11]. The number of possible outcomes yielded by the projective measurement is N (in contrast with N^2 in teleportation) and the unitaries Bob need to implement form a commutative subgroup of the Weyl group of teleportation, namely, it is the cyclic group $\mathbb{Z}_N$. The ensemble of states that can be prepared is an $N-1$ dimensional real manifold. Since an arbitrary state is given by twice as many parameters [$2(N-1)$ real numbers], this kind of RSP realizes remote preparation of "$\frac{1}{2}$ qubit" using 1 ebit of entanglement and 1 bit of classical communication as resources. It is also called *minimum* RSP.

The protocols mentioned so far deal with finite dimensional quantum systems. Nevertheless, light pulses used for quantum communication are essentially described by continuous variables (CV), and CV quantum information processing provides an interesting alternative to the traditional qubit-based approach [12, 13, 14, 15, 16].

In CV teleportation [17, 18, 19], the nonlocal resource shared by Alice and Bob is the Einstein–Podolsky–Rosen state [20] with perfect correlation in both position and momentum. In a quantum optical context, such a correlation can be approximated with a highly squeezed two-mode state of the electromagnetic (EM) field with quadrature amplitudes of the field playing the role of position and momentum. Bell measurement at Alice's site is changed to the simultaneous measurement of the center-of-mass position $\hat{X} \equiv \hat{X}_1 + \hat{X}_{\text{in}}$ and relative momentum $\hat{P} \equiv \hat{P}_1 - \hat{P}_{\text{in}}$ of Alice's half of the EPR pair and the input particle. Finally, the unitary operations Bob has to implement are the phase space displacement operators $\hat{D}(\alpha)$ with $\alpha = X + iP$ obtained from the measurement result.

They form the well-known Heisenberg–Weyl group.

Though CV systems play an important role in quantum information [12, 13, 14, 15, 16, 17, 18, 19], a comprehensive study of continuous variable remote state preparation (CVRSP) is still missing. There are many quantum state engineering schemes based on conditional measurements on one of two entangled light beams. However, they are not RSP, since they are probabilistic. And it is an essential feature of RSP that Bob's action makes it deterministic. The scheme of Ref. [21] can remotely prepare a squeezed state using homodyne detection and it can be made deterministic by Bob applying a phase space displacement operation conditioned on Alice measurement result. Still, only a restricted ensemble of states can be prepared, since Alice has control over one (complex) parameter only.

In the present paper, we will show that by allowing Alice to perform other kinds of measurements, we can enlarge the ensemble of preparable states while letting Bob use the same set of unitaries and keeping the classical communication cost. The ensemble will be parameterized by not only one but continuously many real numbers, that is, by real functions. We will present two other CVRSP protocols involving photon counting and phase measurements that are based on the minimum RSP scheme of [9, 10] by taking the infinite limit in the dimensionality of the state spaces.

The paper is organized as follows. In Sec. II, exact deterministic oblivious remote state preparation of finite dimensional systems with minimal classical communication is introduced. We characterize the non-maximally entangled resource by antilinear operators. Then we give the ensemble in the N dimensional case that can be remotely prepared if Bob applies a special set of unitary operations, namely, the group $\mathbb{Z}_N$ of integer addition modulo N. Sec. III presents our main idea, we extend our analysis to CVRSP, a topic in quantum information processing that has not yet been thoroughly investigated. We present three example protocols, one based on quadrature amplitude measurement, the others involving measurement of the optical phase and the number operator. Sec. IV summarizes our results.

II. MINIMUM RSP IN FINITE DIMENSION

The RSP protocol introduced in Ref. [8] consists of two steps: (i) Alice performs a projective measurement on her half of the shared entangled pair according to the *target* state that she wants to prepare remotely at Bob's site, and communicates its result to Bob; (ii) Bob applies a unitary transformation on his half according to Alice's message to restore the target state. Initially, they share a pure (but not necessarily maximally) entangled state

$$|\Psi\rangle_{AB} = \sum_{k=0}^{N-1} \alpha_k |k\rangle_A |k\rangle_B \qquad (1)$$

where α_k and $|k\rangle$ are the Schmidt coefficients and the corresponding Schmidt vectors. It is very intuitive to view the entangled nature of the two subsystems through the antilinear (conjugate linear) operator

$$R\colon \mathcal{H}_A \to \mathcal{H}_B, \quad R|\phi\rangle_A \equiv {}_A\langle\phi|\Psi\rangle_{AB}, \qquad (2)$$

where the partial scalar product is antilinear in its first argument. To illuminate its physical motivation, suppose that Alice finds her part in a state $|\phi\rangle_A$ after a projective (von Neumann) measurement. Because of entanglement, Bob's state, conditioned on this measurement outcome, reads

$$|\psi\rangle_B = \frac{1}{\sqrt{p}} {}_A\langle\phi|\Psi\rangle_{AB} = \frac{1}{\sqrt{p}} R|\phi\rangle_A, \qquad (3)$$

where the normalizing factor is obtained from the probability p of this measurement event

$$p = \|{}_A\langle\phi|\Psi\rangle_{AB}\|^2 = \|R|\phi\rangle_A\|^2. \qquad (4)$$

The entangled state $|\Psi\rangle_{AB}$ is completely given by the antilinear operator (2) that is mapping a possible measurement eigenstate in system A to the corresponding state of system B after the measurement. This isomorphism between pure entangled states and antilinear operators, and its application to quantum teleportation has been thoroughly investigated in Refs. [22, 23, 24]. Note that the polar decomposition of the antilinear operator (2) is

$$R = \sqrt{\rho_B} J, \qquad (5)$$

where ρ_B is the reduced density operator of the entangled resource and $J\colon \mathcal{H}_A \to \mathcal{H}_B$ is the antiunitary isomorphism that maps Schmidt vectors of system A to those of system B.

Now, *any* target state $|\psi\rangle_B$ can be prepared remotely in a probabilistic way if there is a state $|\phi\rangle_A$ in $\mathcal{H}_A$ that satisfies (3) and Alice manages to project her system onto this state. However, the ensemble $\mathcal{E}$ of target states that can be prepared in a deterministic exact way is rather restricted due to the fact that Bob can apply only a given set of unitary operations U_j on his system in case Alice's measurement fails to result the state $|\phi\rangle_A$. It is easy to see [9, 10, 11] that RSP is possible in the special case when

(i) Bob's unitaries commute with the partial density operator of the entangled resource, i.e.,

$$[\rho_B, U_j] = 0 \quad \text{for all } j \text{ and} \qquad (6)$$

(ii) Alice's measurement eigenstates are obtained from $|\phi\rangle_A$ by the same unitaries as Bob's, more precisely,

$$|\phi_j\rangle_A = J^\dagger U_j^\dagger J |\phi\rangle_A. \qquad (7)$$

where J is the antiunitary isomorphism between $\mathcal{H}_A$ and $\mathcal{H}_B$ introduced in (5). Indeed, if Alice's measurement results the state (7) then Bob's state becomes

$$
\begin{aligned}
|\psi_j\rangle_B &= \frac{1}{\sqrt{p}} R|\phi_j\rangle_A = \frac{1}{\sqrt{p}} \left(\sqrt{\rho_B} J\right) \left(J^\dagger U_j^\dagger J|\phi\rangle_A\right) \\
&= U_j^\dagger \left(\frac{1}{\sqrt{p}} R|\phi\rangle_A\right) = U_j^\dagger |\psi\rangle_B \quad (8)
\end{aligned}
$$

where the probability p must be the same for every measurement outcome (i.e., $p = 1/N$). After he received the classical information about the result, Bob performs the unitary transformation U_j that just turns (8) to the correct target state $|\psi\rangle_B$.

We give an example based on Refs. [9, 10, 11] that satisfies the sufficient RSP conditions (6) and (7). Let us suppose that Bob's set of unitary operations are given by

$$
U_j|k\rangle = e^{2\pi i jk/N}|k\rangle \quad (9)
$$

with $j, k = 0, 1, \ldots, N-1$. They form the cyclic group $\mathbb{Z}_N$ that is a subgroup of the Weyl group used for teleportation. Condition (6) is immediately fulfilled if the eigenstates $|k\rangle_B$ of the unitaries are chosen to be Schmidt vectors of the entangled state (1). It is easy to see that the measurement eigenstates (7) constitute an orthonormal basis if and only if $|\phi\rangle_A$ is an equal-weighted superposition of all the Schmidt vectors $|k\rangle_A$:

$$
|\phi\rangle_A = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-i\varphi_k}|k\rangle_A \quad (10)
$$

where φ_k are the free RSP parameters that are known to Alice but unknown to Bob and, because of the irrelevant global phase factor, $N-1$ of them are independent. Substituting it into Eq. (3), we obtain the ensemble of preparable target states

$$
\mathcal{E} = \left\{ |\psi\rangle = \sum_{k=0}^{N-1} \alpha_k e^{i\varphi_k}|k\rangle \,\middle|\, \varphi_k \in \mathbb{R} \right\} \quad (11)
$$

which is the N dimensional generalization of the equatorial ensemble of Ref. [8]. All the measurement outcomes are equally probable, that is, the probability distribution is uniform, and Bob gains no information about the target state from the classical message.

For a specific element $|\psi\rangle_B$ in (11), the corresponding measurement basis writes

$$
|\phi_j\rangle_A = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N - i\varphi_k}|k\rangle_A. \quad (12)
$$

So Alice uses the parameters φ_k to tune her measuring apparatus according to different target states. A practical way for her to do it is to perform the measurement always in the same fixed basis $|\phi_j^{(0)}\rangle_A$ after she applied

the unitary *pre-measurement* transformation V that explicitly depends on the parameters φ_k and generates the basis (12) appropriately. With our special setup it is convenient to choose

$$
|\phi_j\rangle_A = V^\dagger |\phi_j^{(0)}\rangle_A, \quad (13)
$$

$$
|\phi_j^{(0)}\rangle_A = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N}|k\rangle_A, \quad (14)
$$

$$
V = \sum_{k=0}^{N-1} e^{i\varphi_k}|k\rangle_{AA}\langle k|. \quad (15)
$$

When constructing an RSP setup, the goal is to find an implementation of the pre-measurement transformation and to perform the fixed measurement that allow deterministic exact RSP of the ensemble (11).

III. CONTINUOUS VARIABLE REMOTE STATE PREPARATION

In this section, we extend the idea of Sec. II to continuous variables and present example schemes for CVRSP. We also investigate whether the proposed schemes are practical.

We have considered RSP schemes having uniform probability distribution of the measurement outcome. In the infinite dimensional case with well-normed (physical) entangled state and discrete variable measurement (i.e., one having countable infinitely many outcomes, like photon counting in a mode of the EM field), a discrete probability distribution cannot be uniform because probabilities should sum up to 1. To find a scheme with uniform probability distribution, either the requirement for a well-normed physical entangled state should be removed or continuous variable measurement should be allowed, or both. We shell show examples for all the three cases.

A. RSP by quadrature measurement

Let us consider an ideally correlated EPR pair [20], the same as in ideal CV teleportation. The wave function of this unphysical state is

$$
\Psi(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x_1 - x_2)p}\, dp = \delta(x_1 - x_2). \quad (16)
$$

As a generalization of minimum RSP to systems of continuous variables, one may suppose that Alice measures the momentum of her particle. Since the total momentum is zero, a measurement result of p would imply that Bob's particle is in a momentum eigenstate with momentum $-p$. Therefore, if Alice wanted to remotely prepare the momentum eigenstate $|p_0\rangle_B$ at Bob's side, she just has to message Bob to apply a phase space displacement $\hat{D}(\alpha)$ on his particle with $\alpha = 0 + i(p_0 + p)$. The message consists of one real number $(p_0 + p)$ which yields no

information about the target state $|p_0\rangle_B$ as the probability distribution of the outcome p is uniform. The set of unitaries Bob may need to apply is a subgroup of the Heisenberg–Weyl group of continuous teleportation and is isomorphic to the additive group of real numbers $\mathbb{R}$. Noise and detector inefficiencies in this scheme can be taken into account according to Ref. [21]. However, the ensemble of states that can be prepared this way is rather restricted. The only parameter Alice can control is the real number p_0 and the protocol would draw the same results with a classically correlated shared resource. It is merely a CV reformulation of the Vernam cipher (one-time pad) that is a classical cryptographic protocol using shared randomness as common secret key.

In analogy with Sec. II and Eq. (15), we may let Alice apply a pre-measurement transformation

$$\hat{V} = e^{i\varphi(\hat{X}_1)}, \qquad (17)$$

where now the real function $\varphi \colon \mathbb{R} \to \mathbb{R}$ plays the role of RSP parameters that are under Alice's control but are unknown to Bob. This transforms the entangled state into

$$(\hat{V}\Psi)(x_1, x_2) = e^{i\varphi(x_1)}\delta(x_1 - x_2). \qquad (18)$$

Then Alice performs a momentum measurement. The eigenstate of the momentum operator $\hat{P}_1 = -i\partial_{x_1}$ corresponding to the eigenvalue p is given by the wave function $\phi_p^{(0)}(x_1) = e^{ipx_1}$. If the outcome p occurs, Bob's conditional state will be given by the partial inner product $|\psi\rangle_B = {}_A\langle\phi_p^{(0)}|\hat{V}|\Psi\rangle_{AB}$

$$\psi_p(x_2) = \int_{-\infty}^{\infty} \phi_p^*(x_1)\Psi(x_1, x_2)\,dx_1 = e^{i[\varphi(x_2) - px_2]}. \quad (19)$$

Bob is provided by the momentum displacement operators $\hat{D}_2(ip) = \exp(ip\hat{X}_2)$ with the real parameter p obtained from the classical message. Applying the corresponding operation, he obtains the correct target state

$$\psi(x_2) = e^{i\varphi(x_2)} \qquad (20)$$

that no longer depends on the measurement outcome. We conclude that the ensemble of target states that can be prepared using the present method is

$$\mathcal{E} = \left\{ \psi(x) = e^{i\varphi(x)} \,\Big|\, \varphi \colon \mathbb{R} \to \mathbb{R} \right\}. \qquad (21)$$

Note that the role of position and momentum can be interchanged, and any quadrature operators $\hat{Q}_\theta$ and $\hat{Q}_{\theta+\pi/4}$ can be used instead of them.

We also remark that the continuity of the target wave function (20) should be ensured by the process that realizes the pre-measurement transformation (17). An example in quantum optics for a possible process is the phase displacement operation $\hat{D}(\alpha)$ itself. It can be realized in a homodyne interferometric setup by mixing the input

mode with an intense coherent laser beam on a low reflectivity beam splitter. Since the displacement parameter α can be tuned by adjusting the phase and amplitude of the laser field, one can realize the momentum displacement operator $\hat{D}(ip) = \exp(ip\hat{X})$ corresponding to a linear $\varphi(x) = px$. However, this process alone would lead to the CV Vernam cipher.

More general pre-measurement transformations are provided by CV quantum computation. It can be shown [15] that the transformation $\hat{V} = \exp i(\alpha\hat{X} + \beta\hat{X}^2)$, whose generator is a quadratic polynomial in $\hat{X}$, can be implemented using appropriate combination of phase space displacements and squeezing. Moreover, the cubic phase gate $V = \exp i\gamma\hat{X}^3$ can be realized by combining linear optics with the nonlinear photon number measurement process [13, 14]. As long as $\hat{V}$ commutes with $\hat{X}$, more sophisticated [12] pre-measurement transformations give rise to larger ensemble of preparable states.

To summarize, the proposed CVRSP scheme involves momentum (quadrature) measurement at Alice's side and phase space displacement operations at Bob's side. Remote preparation of the ensemble (21) is possible using the original EPR correlated resource and the communication of one real number.

B. RSP by phase measurement

This example is motivated by the fact that Alice's measurement eigenstates given in Eq. (14) are analogous to the Pegg–Barnett phase states in an N dimensional truncated Hilbert space [25]. Consider two modes of the EM field. We identify states $|k\rangle_A$ and $|k\rangle_B$ defined in Sec. II with photon number states $|n\rangle_A$ and $|n\rangle_B$ of the corresponding field modes and suppose that Bob can apply simple optical phase shifts

$$\hat{U}_\vartheta = \sum_{n=0}^{\infty} e^{i\vartheta n}|n\rangle_{BB}\langle n| \equiv e^{i\vartheta\hat{N}} \qquad (22)$$

on his mode with phase angles $\vartheta = 2\pi j/N$ (for $j = 0$, $\ldots$, $N-1$). We can let Bob take the limit $N \to \infty$ independently of Alice and suppose that he can shift the phase by an arbitrary amount. Thus, in the currently proposed CVRSP setup, the set of Bob's recovering unitary operations is parameterized by a continuous phase variable, that is, it is isomorphic to the group $SU(1)$.

According to Eq. (6), we must choose the entangled shared resource so that its Schmidt vectors are photon number states. Although any such resource would do, it is straightforward to consider a two-mode squeezed vacuum, expressed in the discrete photon number basis as

$$|\mathrm{sq}\rangle_{AB} = \frac{1}{\cosh r}\sum_{n=0}^{\infty}(\tanh r)^n|n\rangle_A|n\rangle_B. \qquad (23)$$

Also, we have that Alice's pre-measuring unitary operation is a phase shift depending on the photon number

$$\hat{V} = \sum_{n=0}^{\infty} e^{i\varphi_n} |n\rangle_{AA}\langle n| \equiv e^{i\varphi(\hat{N})}, \qquad (24)$$

where φ_n or $\varphi(n)$ now denotes a series of real numbers. Theoretically, the angles φ_n of the phase shift can be arbitrary. They are the free RSP parameters that are chosen by Alice but Bob is unaware of them. In practice, however, their choice is rather restricted. We may consider an optical phase shifter as in Eq. (22) for which $\varphi_n = \vartheta n$ is linear in n, or a Kerr nonlinear medium [12] which typically realizes a shift $\varphi_n = \chi n^2$ of the quantum mechanical phase that is quadratic in the photon number n. We can also combine different methods to achieve more complex functions like $\varphi_n = \chi n^2 + \vartheta n$.

Let Alice measure the N dimensional truncated Pegg–Barnett phase states [25] in analogy with Eq. (14)

$$|\vartheta_j\rangle_A = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{2\pi i j n/N} |n\rangle_A. \qquad (25)$$

If Alice communicates the result of her measurement, i.e., the angle $\vartheta_j = 2\pi j/N$ to Bob and he applies the corresponding phase shift (22), the state prepared becomes

$$|\psi\rangle_B = \frac{1}{\cosh r \sqrt{pN}} \sum_{n=0}^{N-1} (\tanh r)^n e^{i\varphi_n} |n\rangle_B. \qquad (26)$$

Finally, in accordance with the Pegg–Barnett principle, we take the limit $N \to \infty$ at the very end of the procedure, after the output state was calculated. The ensemble of states that can be prepared this way is

$$\mathcal{E} = \left\{ |\psi\rangle = \sum_{n=0}^{\infty} \frac{(\tanh r)^n}{\cosh r} e^{i\varphi_n} |n\rangle \,\middle|\, \varphi_n \in \mathbb{R} \right\}. \qquad (27)$$

The squeezing parameter r is fixed in the protocol, while the real numbers φ_n are under Alice's control and unknown to Bob. The transmitted classical information consists of one real number, the angle ϑ.

Note that the measurement of the truncated Pegg–Barnett phase states (25) does not always yield a truncated phase eigenstate. It is because they form a complete basis only in the truncated Hilbert space, and the shared resource (23) has components of photon numbers in mode A higher than $N-1$. Therefore, some of the possible outcomes of an experiment should be discarded. The total probability of success is

$$P = Np = \sum_{n=0}^{N-1} \frac{\tanh^{2n} r}{\cosh^2 r} = 1 - \tanh^{2N} r \qquad (28)$$

that tends to 1 as N increases, that is, the probability of failure decreases exponentially with N.

Recently, schemes have been developed to experimentally perform the direct single-shot measurement of truncated phase eigenstates in a probabilistic way using beam splitters, mirrors, phase shifters and photodetectors [26, 27, 28, 29]. These setups involve N auxiliary modes and the number of the optical elements required scales polynomially with N. Although, they realize POVM measurements, the state projections of our needs can be obtained. However, the probability of success decreases with N. Therefore, at present state of art, an experimental realization of our CVRSP scheme may be applicable for preparation of states not containing too large photon numbers.

To summarize, the proposed CVRSP scheme utilizes two-mode squeezed vacuum as shared entangled resource. Alice performs a unitary pre-measurement operation shifting the phase by an amount depending on the photon number. This can be (but not restricted to), e.g., either or both (i) a constant phase shift ϑ on each photon or (ii) a Kerr-like nonlinear phase shift $\varphi_n = \chi n^2$. Then she performs a phase measurement, she communicates the result to Bob, and he applies a linear phase shift on his mode accordingly. The ensemble of states that can be prepared this way is given by Eq. (27).

C. RSP by photon counting

Now we present a scheme that involves discrete photon number measurement. According to Sec. III, we release the requirement for a well-normed entangled state and consider the following unphysical state

$$|\Psi\rangle_{AB} = \sum_{n=0}^{\infty} |n\rangle_A |n\rangle_B. \qquad (29)$$

As the first step, suppose that Alice wants to remotely prepare the photon number state $|m\rangle_B$. For this purpose, she applies the non-unitary down-shift operator

$$\hat{U}_m = \sum_{n=0}^{\infty} |n\rangle\langle n+m| \equiv [(\hat{n}+1)^{-1/2}\hat{a}]^m \equiv e^{im\hat{\Phi}} \qquad (30)$$

on system A that decreases the photon number by m dropping out all components with number of photons less than m. Note that $\hat{U}$ is a simple exponential function of the non-Hermitian phase operator $\hat{\Phi}$. This way, Alice turns the shared state into

$$|\Psi'\rangle_{AB} = \sum_{n=m}^{\infty} |n-m\rangle_A |n\rangle_B = \sum_{n=0}^{\infty} |n\rangle_A |n+m\rangle_B. \qquad (31)$$

Now she performs a photon counting measurement and communicates the result n to Bob. Since his conditional state is $|n+m\rangle_B$, he just has to apply the down-shift operation $\hat{U}_n$ to obtain the correct target state $|m\rangle_B$.

Though the above example is a simple infinite dimensional extension of the classical Vernam cipher, we will

show in a rigorous way that, by letting Alice apply more complex pre-measurement operations instead of (30), we can enlarge the ensemble of preparable states and turn the scheme into a CVRSP-like protocol.

Let us consider first the N dimensional truncated entangled resource

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |\vartheta_j\rangle_A |-\vartheta_j\rangle_B = \frac{1}{\sqrt{N}} \sum_{n=1}^{N-1} |n\rangle_A |n\rangle_B,$$

(32)

where we used the shorthand $-\vartheta_j$ for $\vartheta_{N-j} = 2\pi - \vartheta_j$. The total phase of the two modes and the relative photon number are both zero. Now we chose Bob's unitaries to have the phase states $|\vartheta_j\rangle_B$ as eigenstates

$$U_n|\vartheta_j\rangle_B = e^{2\pi ijn/N}|\vartheta_j\rangle_B,$$
$$U_n|k\rangle_B = |k \ominus n\rangle_B,$$

(33)

If Φ denotes the Hermitian phase operator in the N dimensional truncated Hilbert space then U_1 corresponds to $\exp(i\Phi)$, and $U_n = \exp(in\Phi)$ and there is no problem with the unitarity of U_n.

We define Alice's pre-measuring unitary operation as

$$V = \sum_{j=0}^{N-1} e^{i\varphi_j}|\vartheta_j\rangle_{AA}\langle\vartheta_j| \equiv e^{i\varphi(\Phi)} = \sum_{n=0}^{N-1} f_n U_n, \quad (34)$$

where the coefficients f_n are the well-normed discrete Fourier series of the phase shifts $e^{i\varphi_j}$

$$f_n = \frac{1}{N} \sum_{j=0}^{N-1} e^{-2\pi ijn/N} e^{i\varphi_j}, \qquad \sum_{n=0}^{N-1} |f_n|^2 = 1. \quad (35)$$

Let Alice perform a photon counting measurement with eigenstates $|n\rangle_A$. After the measurement and the corresponding unitary transformation U_n we obtain that the following state is prepared remotely at Bob's site

$$|\psi\rangle_B = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{i\varphi_j}|-\vartheta_j\rangle_B = \sum_{n=0}^{N-1} f_n|n\rangle_B. \quad (36)$$

In the limit $N \to \infty$, we can use the unnormed continuous phase eigenbasis

$$|\vartheta\rangle = \sum_{n=0}^{\infty} e^{i\vartheta n}|n\rangle \quad (37)$$

to express the unphysical maximally entangled EPR state

$$|\Psi\rangle_{AB} = \frac{1}{2\pi} \int_0^{2\pi} |\vartheta\rangle_A |-\vartheta\rangle_B \, d\vartheta = \sum_{n=0}^{\infty} |n\rangle_A |n\rangle_B. \quad (38)$$

U_1 turns into the non-unitary $\hat{U}_1 = \exp(i\hat{\Phi}) = (\hat{n}+1)^{-1/2}\hat{a}$ photon number down-shift operator, and the

Fourier series (35) turn into the discrete Fourier transform of the periodic phase shift function $e^{i\varphi(\vartheta)}$

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\vartheta} e^{i\varphi(\vartheta)} \, d\vartheta, \qquad \sum_{n=0}^{\infty} |f_n|^2 = 1. \quad (39)$$

Alice's non-unitary pre-measuring operator becomes

$$\hat{V} = \int_0^{2\pi} e^{i\varphi(\vartheta)}|\vartheta\rangle_{AA}\langle\vartheta| \, d\vartheta \equiv e^{i\varphi(\hat{\Phi})}$$
$$= f_0\hat{\mathbb{1}} + \sum_{n=1}^{\infty} \left(f_n\hat{U}_n + f_{-n}\hat{U}_n^\dagger\right). \quad (40)$$

After Alice applied this pre-measurement transformation, she performs a photon counting measurement. If she finds n photons in mode A, then Bob's conditional state becomes

$$|\psi_n\rangle_B = \sum_{k=0}^{\infty} f_{(k-n)}|k\rangle_B = \sum_{k=-n}^{\infty} f_k|k+n\rangle_B. \quad (41)$$

Applying the down-shift operator $\hat{U}_n$, we obtain the target state that has now become unconditional on the measurement outcome:

$$|\psi\rangle_B = \frac{1}{2\pi} \int_0^{2\pi} e^{i\varphi(\vartheta)}|-\vartheta\rangle_B \, d\vartheta = \sum_{n=0}^{\infty} f_n|n\rangle_B, \quad (42)$$

where the periodic real function $\varphi(\vartheta)$ or the Fourier coefficients f_n that are given by (39) serve as the RSP parameters.

We note that the presented scheme seems to be irrealistic for that it utilizes unphysical shared state and the photon number down-shift operation is hard to realize. The setup is still interesting theoretically because it shows that unitary operations at Bob's site are not necessary, irreversible nonunitary operations can also be used to restore the output state.

IV. CONCLUSIONS

We have considered exact deterministic remote state preparation with minimal classical communication. For finite N dimensional systems, this corresponds to a projective measurement at Alice's side, communication of the minimal amount of $\log N$ bits of classical information, and a unitary transformation at Bob's side. The ensemble of preparable states is parameterized by $N-1$ real numbers called RSP parameters which are under Alice's control and unknown to Bob.

We have introduced continuous variable versions of the above remote state preparation analogous to continuous variable teleportation. While in the latter, two real numbers are messaged, the classical communication cost of our CVRSP schemes is one real number only. Still,

there are infinitely many RSP parameters that determine the target state. We have proposed three example setups consisting of (i) momentum measurement and momentum displacement operation, (ii) phase measurement and phase shifters, and (iii) photon number measurement and photon number shift operations. Alice's unitary pre-measurement transformation, into which the RSP parameters are fed, is function of the variable canonically conjugated to what is measured, i.e., it is function of (i) the position operator, (ii) the number operator, and (iii) the non-Hermitian phase operator, respectively.

[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[2] C. H. Bennett, P. Hayden, D. W. Leung, P. W. Shor, and A. Winter, IEEE Trans. Inform. Theory **51**, 56 (2005), quant-ph/0307100.

[3] H. K. Lo, Phys. Rev. A **62**, 012313 (2000).

[4] I. Devetak and T. Berger, Phys. Rev. Lett. **87**, 197901 (2001).

[5] D. W. Leung and P. W. Shor, Phys. Rev. Lett. **90**, 127905 (2003).

[6] A. Hayashi, T. Hashimoto, and M. Horibe, Phys. Rev. A **67**, 052302 (2003).

[7] A. Abeyesinghe and P. Hayden, Phys. Rev. A **68**, 062319 (2003).

[8] A. K. Pati, Phys. Rev. A **63**, 014302 (2000).

[9] B. Zeng and P. Zhang, Phys. Rev. A **65**, 022316 (2002).

[10] M.-Y. Ye, Y.-S. Zhang, and G.-C. Guo, Phys. Rev. A **69**, 022310 (2004).

[11] Z. Kurucz and P. Adam, J. Opt. B: Quantum Semiclass. Opt. **7**, 135 (2005).

[12] S. Lloyd and S. L. Braunstein, Phys. Rev. Lett. **82**, 1784 (1999).

[13] D. Gottesman, A. Kitaev, and J. Preskill, Phys. Rev. A **64**, 012310 (2001).

[14] S. D. Bartlett and B. C. Sanders, Phys. Rev. A **65**, 042304 (2002).

[15] S. D. Bartlett, B. C. Sanders, S. L. Braunstein, and K. Nemoto, Phys. Rev. Lett. **88**, 097904 (2002).

[16] S. D. Bartlett and W. J. Munro, Phys. Rev. Lett. **90**, 117901 (2003).

[17] L. Vaidman, Phys. Rev. A **49**, 1473 (1994).

[18] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).

[19] A. Furusawa, J. L. Sorensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science **282**, 706 (1998).

[20] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[21] M. G. A. Paris, M. Cola, and R. Bonifacio, J. Opt. B: Quantum Semiclass. Opt. **5**, S360 (2003).

[22] A. Uhlmann, in *Lecture Notes on Physics*, edited by A. Borowiec (Springer-Verlag, Berlin, 2000), vol. 539, pp. 93–105, quant-ph/9901027.

[23] Z. Kurucz, M. Koniorczyk, and J. Janszky, Fortschr. Phys. **49**, 1019 (2001), quant-ph/0308020.

[24] Z. Kurucz, M. Koniorczyk, P. Adam, and J. Janszky, J. Opt. B: Quantum Semiclass. Opt. **5**, S627 (2003).

[25] D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665 (1989).

[26] S. M. Barnett and D. T. Pegg, Phys. Rev. Lett. **76**, 4148 (1996).

[27] K. L. Pregnell and D. T. Pegg, Phys. Rev. Lett. **89**, 173601 (2002).

[28] K. L. Pregnell and D. T. Pegg, Phys. Rev. A **67**, 063814 (2003).

[29] X. Zou, K. Pahlke, and W. Mathis, Phys. Lett. A **323**, 329 (2004).